Asymptotics of non-diagonal multiple orthogonal polynomials for a cubic weight

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We study (non-hermitian) multiple orthogonal polynomial $P_{n,m}(z) = z^N + \dots$ defined through

$$\int_{\Sigma_1} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, \dots, n-1,$$
$$\int_{\Sigma_2} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, \dots, m-1,$$

where $N = n + m$ and Σ_1, Σ_2 are contours extending to ∞ with angles $-\frac{2\pi}{3}$, 0 and $-\frac{2\pi}{3}$, $\frac{2\pi}{3}$, respectively. The asymptotic behavior of $P_{n,m}$ as $m, n \to \infty$, in the regime $n/N \to \alpha \in (0, 1/2)$, is governed by vector-valued measures $\vec{\mu}_\alpha$ that provide a saddle point for certain energy functionals in an electrostatic model in which the mutual interaction comprises both attracting and repelling forces. For instance, the sequence of zero counting measures associated with $(P_{n,m})$ converges to the sum of the first two components of $\vec{\mu}_\alpha$.

Measures $\vec{\mu}_\alpha$ can be characterized by a cubic algebraic equation (spectral curve) whose solutions are appropriate combinations of the Cauchy transform of the components of $\vec{\mu}_\alpha$. As a consequence, $\vec{\mu}_\alpha$ are supported on a finite number of analytic arcs, that are trajectories of a quadratic differential globally defined on a three-sheeted Riemann surface. The complete description of the so-called critical graph for such a differential (and its dynamics as a function of the parameter α) is the key ingredient of the Riemann-Hilbert asymptotic analysis of the polynomials $P_{n,m}$.